Even and odd nonlinear charge coherent states and their nonclassical properties

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Abstract

The (over)completeness of even and odd nonlinear charge coherent states is proved and their generation explored. They are demonstrated to be generalized entangled nonlinear coherent states. A D-algebra realization of the $SU_f(1,1)$ generators is given in terms of them. They are shown to exhibit $SU_f(1,1)$ squeezing and two-mode f-antibunching for some particular types of f-nonlinearity, but neither one-mode nor two-mode f-squeezing.

Keywords: Even/odd charge coherent state; f-Deformation; Completeness relation; Entangled coherent state; Squeezing

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1. Introduction

The coherent states introduced by Schrödinger [1] and Glauber [2] are eigenstates of the boson annihilation operator, and have the very broad range of applications in the areas of physics [3−7]. However, in all the cases the quanta involved are uncharged. In 1976, Bhaumik et al. [4,8,9] proposed the boson coherent states which, possessing definite charge, are eigenstates of both the pair boson annihilation operator and the charge operator. This kind of states are the so-called charge coherent states. On the basis of this work, the charge coherent states for SU(2) [10], SU(3) [11], and arbitrary compact Lie groups [12] were later developed.

The notion of charge coherent states has turned out to be very useful in various branches, such as elementary particle physics [9,13−17], quantum field theory [12,18,19], nuclear physics [20], thermodynamics [21−23], quantum mechanics [24], and quantum optics [25−27]. Some proposals for implementations of such states in quantum optics were also put forward [25,26,28−33].

The charge coherent states, also called pair coherent states, are a type of correlated two-mode states, which are not only important for studying fundamental aspects of quantum mechanics, such as quantum nonlocality (Bell-inequality violation) tests [34−36], but also useful in quantum information processing, such as quantum entanglement [37,38] and quantum teleportation [39].

As is well known, the even and odd coherent states [40] are two orthonormalized eigenstates of the square of the boson annihilation operator, and attain an important position in the study of quantum optics [41−43]. Motivated by this idea, one of the authors (X.-M.L.) [44] has extended the charge coherent states to the even and odd

charge coherent states, defined as two orthonormalized eigenstates of both the square of the pair boson annihilation operator and the charge operator.

Quantum groups [45,46], introduced as a mathematical description of deformed Lie algebras, have provided the possibility of extending the concept of coherent states to the case of q-deformations [47−51]. A q-deformed harmonic oscillator [47,52] is represented by the q-boson annihilation and creation operators, which satisfy the quantum (q-deformed) Heisenberg-Weyl algebra [47,52,53], the latter being an elementary object in quantum groups. The q-deformed coherent states advanced by Biedenharn [47] are eigenstates of the q-boson annihilation operator. Such states have been well investigated [48,49,54,55], and used extensively in quantum optics and mathematical physics [51,56−60]. Moreover, the q-deformed charge coherent states [61,62] were introduced as eigenstates of both the pair q-boson annihilation operator and the charge operator.

A straightforward generalization of the q-deformed coherent states is to define the even and odd q-deformed coherent states [63], which are two orthonormalized eigenstates of the square of the q-boson annihilation operator. On a similar route, the authors (X.-M.L. and C.Q.) [64] have extended the q-deformed charge coherent states to the even and odd q-deformed charge coherent states, defined as two orthonormalized eigenstates of both the square of the pair q-boson annihilation operator and the charge operator.

Study of q-deformed harmonic oscillators has shown that these dynamical systems can be interpreted as nonlinear oscillators with specific expotential dependence of the frequency on vibration amplitude [65], thus inspiring the extension of the q-deformations to the f-deformations for which the dependence of frequency on the

amplitude is described by an arbitrary function (called nonlinear function) [66,67]. A f-deformed oscillator (or nonlinear oscillator) was defined by means of the f-deformed annihilation and creation operators, which satisfy the f-deformed algebra (or nonlinear algebra) [66,68−73]. The nonlinear coherent states [66,70,72−74] were already constructed as eigenstates of the f-deformed annihilation operator, and their mathematical properties and nonclassical features discussed in detail [66,74−77]. A class of nonlinear coherent states can be realized physically as the stationary states of the centre-of-mass motion of a trapped ion [74]. In addition, the nonlinear charge coherent states [78,79] were introduced as eigenstates of both the pair f-deformed annihilation operator and the charge operator.

A further development of the nonlinear coherent states is performed by the even and odd nonlinear coherent states [80,81], which are two orthonormalized eigenstates of the square of the f-deformed annihilation operator. In a parallel approach, the nonlinear charge coherent states have been extended to the even and odd nonlinear charge coherent states [82,83], defined as two orthonormalized eigenstates of both the square of the pair f-deformed annihilation operator and the charge operator. In this paper, it is very desirable to prove a completeness relation of the even and odd nonlinear charge coherent states, explore their generation, and study their properties in the aspects of both mathematics and quantum optics.

This paper is arranged as follows. In Section 2, a review of the even and odd nonlinear charge coherent states is presented, and the proof of their completeness relation given. Section 3 is devoted to generating such states. They are used to realize a D-algebra of the $\mathrm{SU}_f(1,1)$ generators in Section 4. Their nonclassical properties, such as $\mathrm{SU}_f(1,1)$ squeezing, single- or two-mode f-squeezing, and two-mode

f-antibunching, are studied in Section 5. Section 6 contains a summary of the results.

2. Completeness of even and odd nonlinear charge coherent states

Two mutually commuting f-deformed oscillators are defined in terms of two pairs of independent f-deformed annihilation and creation operators A_i, $A_i^\dagger$ ($i = 1, 2$), together with corresponding number operators N_i, which are given by

$$A_i = a_i f(N_i) = f(N_i + 1)a_i, \qquad N_i = a_i^\dagger a_i, \tag{1}$$

$$A_i^\dagger = f^\dagger(N_i)a_i^\dagger = a_i^\dagger f^\dagger(N_i + 1), \tag{2}$$

where f is a well behaved operator-valued function of N_i; and a_i and $a_i^\dagger$ are the annihilation and creation operators of usual linear harmonic oscillators, respectively. f is a deformation function and means nonlinearity. The definition of f-deformed oscillators is an extension of the notion of q-deformed harmonic oscillators. The usual harmonic oscillator and q-deformed one are the two special cases of f-deformed oscillators with $f(N_i) = 1$ and $f(N_i) = \sqrt{[N_i]/N_i}$ [84], respectively, where

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{3}$$

with q being a positive real deformation parameter.

The f-deformed algebra commutation relations are

$$[N_i, A_i] = -A_i, \qquad [N_i, A_i^\dagger] = A_i^\dagger, \tag{4}$$

$$[A_i, A_i^\dagger] = (N_i + 1)f^2(N_i + 1) - N_i f^2(N_i), \tag{5}$$

where f is chosen to be real and $f^\dagger(N_i) = f(N_i)$. Of course, the usual Heisenberg-Weyl algebra and q-deformed one are restored when $f(N_i) = 1$ and $f(N_i) = \sqrt{[N_i]/N_i}$, respectively.

Let $|n\rangle_i$ ($n = 0, 1, 2, \ldots$) denote basis states of the Fock space for the mode i, where $|n\rangle_i$ are eigenstates of N_i corresponding to the eigenvalues n. Thus, the operators A_i, $A_i^\dagger$ and N_i can be realized by

$$A_i = \sum_{n=0}^{\infty} \sqrt{n} f(n) |n-1\rangle_{i\ i}\langle n|, \qquad N_i = \sum_{n=0}^{\infty} n |n\rangle_{i\ i}\langle n|, \tag{6}$$

$$A_i^\dagger = \sum_{n=0}^{\infty} \sqrt{n} f(n) |n\rangle_{i\ i}\langle n-1|. \tag{7}$$

We first briefly review the nonlinear charge coherent states and the even (odd) ones. The operators A_1 ($A_1^\dagger$) and A_2 ($A_2^\dagger$) are assigned the "charge" quanta 1 and -1, respectively. Thus the charge operator is given by

$$Q = N_1 - N_2. \tag{8}$$

In view of the fact that

$$[Q, A_1 A_2] = 0, \tag{9}$$

the nonlinear charge coherent states are defined as eigenstates of both the pair f-deformed annihilation operator $A_1 A_2$ ($\equiv a_1 f(N_1) a_2 f(N_2)$) and the charge operator Q, i.e.,

$$A_1 A_2 |\xi, \underline{q}, f\rangle = \xi |\xi, \underline{q}, f\rangle, \qquad Q |\xi, \underline{q}, f\rangle = \underline{q} |\xi, \underline{q}, f\rangle, \tag{10}$$

where ξ is a complex number and $\underline{q}$ is the charge number, which is a fixed integer. Suppose the function $f(n)$ has no zeroes at positive integers. The q-deformed harmonic oscillator is just in this case. With the help of the two-mode Fock space's completeness relation

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |m, n\rangle\langle m, n| = I, \qquad |m, n\rangle \equiv |m\rangle_1 |n\rangle_2, \tag{11}$$

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the nonlinear charge coherent states can be expanded as

$$|\xi, \underline{q}, f\rangle = N_{\underline{q}f} \sum_{p=\max(0,-\underline{q})}^{\infty} \frac{\xi^{p+\min(0,\underline{q})}}{[p!(p+\underline{q})!]^{1/2} f(p)! f(p+\underline{q})!} |p+\underline{q}, p\rangle$$

$$= \begin{cases} N_{\underline{q}f} \sum_{n=0}^{\infty} \frac{\xi^{n}}{[n!(n+\underline{q})!]^{1/2} f(n)! f(n+\underline{q})!} |n+\underline{q}, n\rangle, & \underline{q} \geq 0, \\ N_{\underline{q}f} \sum_{n=0}^{\infty} \frac{\xi^{n}}{[n!(n-\underline{q})!]^{1/2} f(n)! f(n-\underline{q})!} |n, n-\underline{q}\rangle, & \underline{q} \leq 0, \end{cases} \tag{12}$$

where the normalization factor $N_{\underline{q}f}$ is given by

$$N_{\underline{q}f} \equiv N_{\underline{q}f}(|\xi|^2) = \left\{ \sum_{n=0}^{\infty} \frac{(|\xi|^2)^n}{n!(n+|\underline{q}|)![f(n)! f(n+|\underline{q}|)!]^2} \right\}^{-1/2}, \tag{13}$$

and

$$f(n)! = f(n)f(n-1)\cdots f(1), \qquad f(0)! = 1. \tag{14}$$

It is noted that the above nonlinear charge coherent states are somewhat different from those defined in Ref. [78] ([79]), the latter being eigenstates of both the operator $a_1 f_1(N_1) a_2 f_2(N_2)$ $(a_1 a_2 g(N_1, N_2))$ and Q, where f_i is a function of N_i; g is that of N_1 and N_2. The case of $f_1 = f_2 = f$ $(g(N_1, N_2) = f(N_1)f(N_2))$ coincides with the results given above.

The even and odd nonlinear charge coherent states are defined as two orthonormalized eigenstates of both the square $(A_1 A_2)^2$ and the operator Q, i.e.,

$$(A_1 A_2)^2 |\xi, \underline{q}, f\rangle_{e(o)} = \xi^2 |\xi, \underline{q}, f\rangle_{e(o)}, \quad Q|\xi, \underline{q}, f\rangle_{e(o)} = \underline{q}|\xi, \underline{q}, f\rangle_{e(o)}, \quad {}_e\langle \xi, \underline{q}, f | \xi, \underline{q}, f\rangle_o = 0. \tag{15}$$

The solutions to Eq. (15) are

$$|\xi, \underline{q}, f\rangle_e = N_{\underline{q}f}^e \sum_{p=\max(0,-\underline{q}/2)}^{\infty} \frac{\xi^{2p+\min(0,\underline{q})}}{[(2p)!(2p+\underline{q})!]^{1/2} f(2p)! f(2p+\underline{q})!} |2p+\underline{q}, 2p\rangle$$

$$= \begin{cases} N_{\underline{q}f}^e \sum_{n=0}^{\infty} \frac{\xi^{2n}}{[(2n)!(2n+\underline{q})!]^{1/2} f(2n)! f(2n+\underline{q})!} |2n+\underline{q}, 2n\rangle, & \underline{q} \geq 0, \\ N_{\underline{q}f}^e \sum_{n=0}^{\infty} \frac{\xi^{2n}}{[(2n)!(2n-\underline{q})!]^{1/2} f(2n)! f(2n-\underline{q})!} |2n, 2n-\underline{q}\rangle, & \underline{q} \leq 0, \end{cases} \tag{16}$$

$$|\xi, \underline{q}, f\rangle_o$$

$$= N^o_{\underline{q}f} \sum_{p=\max(0,-\underline{q}/2)}^{\infty} \frac{\xi^{2p+1+\min(0,\underline{q})}}{[(2p+1)!(2p+1+\underline{q})!]^{1/2} f(2p+1)! f(2p+1+\underline{q})!} |2p+1+\underline{q}, 2p+1\rangle$$

$$= \begin{cases} N^o_{\underline{q}f} \sum_{n=0}^{\infty} \frac{\xi^{2n+1}}{[(2n+1)!(2n+1+\underline{q})!]^{1/2} f(2n+1)! f(2n+1+\underline{q})!} |2n+1+\underline{q}, 2n+1\rangle, & \underline{q} \geq 0, \\ N^o_{\underline{q}f} \sum_{n=0}^{\infty} \frac{\xi^{2n+1}}{[(2n+1)!(2n+1-\underline{q})!]^{1/2} f(2n+1)! f(2n+1-\underline{q})!} |2n+1, 2n+1-\underline{q}\rangle, & \underline{q} \leq 0, \end{cases} \tag{17}$$

where the normalization factors $N^{e(o)}_{\underline{q}f}$ are given by

$$N^e_{\underline{q}f} \equiv N^e_{\underline{q}f}(|\xi|^2) = \left\{ \sum_{n=0}^{\infty} \frac{(|\xi|^2)^{2n}}{(2n)!(2n+|\underline{q}|)![f(2n)! f(2n+|\underline{q}|)!]^2} \right\}^{-1/2}, \tag{18}$$

$$N^o_{\underline{q}f} \equiv N^o_{\underline{q}f}(|\xi|^2) = \left\{ \sum_{n=0}^{\infty} \frac{(|\xi|^2)^{2n+1}}{(2n+1)!(2n+1+|\underline{q}|)![f(2n+1)! f(2n+1+|\underline{q}|)!]^2} \right\}^{-1/2}. \tag{19}$$

As two special cases, for $f(n) = 1$, they reduce to the usual even and odd charge coherent states constructed by the author (X.-M.L.) [44]; for $f(n) = \sqrt{[n]/n}$, they reduce to the even and odd q-deformed charge coherent states constructed by the authors (X.-M.L. and C.Q.) [64]. Note however that $|\xi, \underline{q}, f\rangle_{e(o)}$ are not eigenstates of $A_1 A_2$.

We should stress that the representation (16) ((17)) of even (odd) nonlinear charge coherent states as bipartite pure states by a biorthogonal sum is a standard form of the Schmidt decomposition [85−89]. Obviously, both even and odd states are entangled states since the Schmidt number for such states is infinite.

It should be remarked that the even (odd) states $|\xi, \underline{q}, f\rangle_{e(o)}$ are normalizable provided $N^{e(o)}_{\underline{q}f}$ are non-zero and finite. This means that the terms in summation for $\left(N^{e(o)}_{\underline{q}f}\right)^{-2}$ should be such that

$$|\xi| < \lim_{n \to \infty} n f^2(n). \tag{20}$$

If $|f(n)|$ decreases faster than $n^{-\frac{1}{2}}$ for large n, then the range of ξ, for which the states $|\xi, \underline{q}, f\rangle_{e(o)}$ are normalizable, is restricted to values satisfying (20) and in other cases the range of ξ is unrestricted.

From (16) and (17), it follows that

$$_{e(o)}\langle \xi, \underline{q}, f | \xi', \underline{q}', f \rangle_{e(o)} = N_{\underline{q}f}^{e(o)}(|\xi|^2) N_{\underline{q}f}^{e(o)}(|\xi'|^2) \left[N_{\underline{q}f}^{e(o)}(\xi^* \xi') \right]^{-2} \delta_{\underline{q},\underline{q}'}, \qquad (21)$$

$$_{e}\langle \xi, \underline{q}, f | \xi', \underline{q}', f \rangle_{o} = 0. \qquad (22)$$

This further indicates that the even (odd) nonlinear charge coherent states are orthogonal with respect to the charge number $\underline{q}$ and that arbitrary even and odd nonlinear charge coherent states are orthogonal to each other. However, these even (odd) states are nonorthogonal with respect to the parameter ξ.

For the mean values of the operators N_1 and N_2, there exists the relation

$$_{e(o)}\langle \xi, \underline{q}, f | N_1 | \xi, \underline{q}, f \rangle_{e(o)} = \underline{q} + {}_{e(o)}\langle \xi, \underline{q}, f | N_2 | \xi, \underline{q}, f \rangle_{e(o)}. \qquad (23)$$

In terms of the even and odd nonlinear charge coherent states, the nonlinear charge coherent states can be expanded as

$$|\xi, \underline{q}, f\rangle = N_{\underline{q}f} \left[(N_{\underline{q}f}^e)^{-1} |\xi, \underline{q}, f\rangle_e + (N_{\underline{q}f}^o)^{-1} |\xi, \underline{q}, f\rangle_o \right], \qquad (24)$$

where the normalization factors are such that

$$N_{\underline{q}f}^{-2} = (N_{\underline{q}f}^e)^{-2} + (N_{\underline{q}f}^o)^{-2}. \qquad (25)$$

To make up a completeness relation of the even and odd nonlinear charge coherent states, we introduce another type of even and odd nonlinear charge coherent states $|\xi, \underline{q}, f\rangle\rangle_{e(o)}$, which are two orthonormalized eigenstates of both the square $(\tilde{A}_1 \tilde{A}_2)^2$ and the operator Q, where

$$\tilde{A}_i = a_i \frac{1}{f(N_i)}. \qquad (26)$$

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It is evident that the representations of $|\xi, \underline{q}, f\rangle\rangle_{e(o)}$ in the two-mode Fock space are the same as $|\xi, \underline{q}, f\rangle_{e(o)}$ given in Eqs. (16) and (17) except for replacing $f(n)$ by $\frac{1}{f(n)}$.

We now prove that the even and odd nonlinear charge coherent states form an (over)complete set, that is to say

$$\sum_{\underline{q}=-\infty}^{\infty} \int \frac{d^2\xi}{\pi} 2|\xi|^{|\underline{q}|} K_{\underline{q}}(2|\xi|) \left[(N_{\underline{q}f}^e N_{\underline{q}\frac{1}{f}}^e)^{-1} |\xi, \underline{q}, f\rangle_e {}_e\langle\langle\xi, \underline{q}, f| + (N_{\underline{q}f}^o N_{\underline{q}\frac{1}{f}}^o)^{-1} |\xi, \underline{q}, f\rangle_o {}_o\langle\langle\xi, \underline{q}, f| \right]$$

$$\equiv \sum_{\underline{q}=-\infty}^{\infty} I_{\underline{q}} = I, \tag{27}$$

where $K_{\underline{q}}(Z)$ is the second kind of modified Bessel function of order $\underline{q}$ [90], and $N_{\underline{q}\frac{1}{f}}^{e(o)}$ is a normalization factor of $|\xi, \underline{q}, f\rangle\rangle_{e(o)}$.

In fact, for $\underline{q} \geq 0$, we have

$$
\begin{aligned}
I_{\underline{q}} &= \int \frac{d^2\xi}{\pi} 2|\xi|^{\underline{q}} K_{\underline{q}}(2|\xi|) \sum_{j=0}^{1} \sum_{n,m} \xi^{2n+j} \xi^{*2m+j} \\
&\quad \times \frac{f(2m+j)! f(2m+j+\underline{q})! |2n+j+\underline{q}, 2n+j\rangle\langle 2m+j+\underline{q}, 2m+j|}{\left[(2n+j)!(2n+j+\underline{q})!(2m+j)!(2m+j+\underline{q})! \right]^{1/2} f(2n+j)! f(2n+j+\underline{q})!} \\
&= \int_0^{\infty} \frac{d|\xi|}{\pi} 2|\xi|^{\underline{q}+1} K_{\underline{q}}(2|\xi|) \sum_{j=0}^{1} \sum_{n,m} |\xi|^{2(n+m+j)} \int_{-\pi}^{\pi} d\theta e^{2i(n-m)\theta} \\
&\quad \times \frac{f(2m+j)! f(2m+j+\underline{q})! |2n+j+\underline{q}, 2n+j\rangle\langle 2m+j+\underline{q}, 2m+j|}{\left[(2n+j)!(2n+j+\underline{q})!(2m+j)!(2m+j+\underline{q})! \right]^{1/2} f(2n+j)! f(2n+j+\underline{q})!} \\
&= \sum_{j=0}^{1} \sum_{n=0}^{\infty} \frac{4|2n+j+\underline{q}, 2n+j\rangle\langle 2n+j+\underline{q}, 2n+j|}{(2n+j)!(2n+j+\underline{q})!} \int_0^{\infty} d|\xi||\xi|^{4n+2j+\underline{q}+1} K_{\underline{q}}(2|\xi|) \\
&= \sum_{j=0}^{1} I_{\underline{q}}^j \\
&= \sum_{n=0}^{\infty} |n+\underline{q}, n\rangle\langle n+\underline{q}, n|, \tag{28}
\end{aligned}
$$

where

$$I_{\underline{q}}^j = \sum_{n=0}^{\infty} |2n+j+\underline{q}, 2n+j\rangle\langle 2n+j+\underline{q}, 2n+j|, \quad j = 0, 1. \tag{29}$$

Similarly, for $\underline{q} \leq 0$, we get

$$I_{\underline{q}} = \sum_{j=0}^{1} I_{\underline{q}}^{j}$$

$$= \sum_{n=0}^{\infty} |n, n - \underline{q}\rangle \langle n, n - \underline{q}|, \tag{30}$$

where

$$I_{\underline{q}}^{j} = \sum_{n=0}^{\infty} |2n + j, 2n + j - \underline{q}\rangle \langle 2n + j, 2n + j - \underline{q}|, \quad j = 0, 1. \tag{31}$$

Consequently, we derive

$$\sum_{\underline{q}=-\infty}^{\infty} I_{\underline{q}} = \sum_{n=0}^{\infty} \left(\sum_{\underline{q}=-\infty}^{-1} |n, n - \underline{q}\rangle \langle n, n - \underline{q}| + \sum_{\underline{q}=0}^{\infty} |n + \underline{q}, n\rangle \langle n + \underline{q}, n| \right)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |m, n\rangle \langle m, n| = I. \tag{32}$$

Hence, the even and odd nonlinear charge coherent states are qualified to make up an overcomplete representation. It should be mentioned that $I_{\underline{q}}$ represents the resolution of unity in the subspace where $Q = \underline{q}$; $I_{\underline{q}}^{e(o)} (\equiv I_{\underline{q}}^{0(1)})$ represents that in the even (odd) subspace where $Q = \underline{q}$ and satisfies

$$I_{\underline{q}}^{j} I_{\underline{q}'}^{j'} = I_{\underline{q}}^{j} \delta_{j,j'} \delta_{\underline{q},\underline{q}'}. \tag{33}$$

For Eq. (27), its conjugate is

$$\sum_{\underline{q}=-\infty}^{\infty} \int \frac{d^2\xi}{\pi} 2|\xi|^{|\underline{q}|} K_{\underline{q}}(2|\xi|) \left[(N_{\underline{q}f}^{e} N_{\underline{q}\frac{1}{f}}^{e})^{-1} |\xi, \underline{q}, f\rangle\rangle_{e\ e}\langle\xi, \underline{q}, f| + (N_{\underline{q}f}^{o} N_{\underline{q}\frac{1}{f}}^{o})^{-1} |\xi, \underline{q}, f\rangle\rangle_{o\ o}\langle\xi, \underline{q}, f| \right]$$

$$\equiv \sum_{\underline{q}=-\infty}^{\infty} I_{\underline{q}} = I. \tag{34}$$

Note that in (27) the ket and bra are not mutually Hermite conjugate.

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3. Generation of even and odd nonlinear charge coherent states

From (12), (16) and (17), it follows that [82,83]

$$|\xi, \underline{q}, f\rangle_e = \frac{1}{2} \frac{N_{\underline{q}f}^e}{N_{\underline{q}f}} (|\xi, \underline{q}, f\rangle + |-\xi, \underline{q}, f\rangle), \tag{35}$$

$$|\xi, \underline{q}, f\rangle_o = \frac{1}{2} \frac{N_{\underline{q}f}^o}{N_{\underline{q}f}} (|\xi, \underline{q}, f\rangle - |-\xi, \underline{q}, f\rangle). \tag{36}$$

This shows that the even (odd) nonlinear charge coherent states can be obtained by the symmetric (antisymmetric) combination of nonlinear charge coherent states as the charge is conserved. This is similar to the case of even (odd) nonlinear coherent states, which are combinations of nonlinear coherent states, namely,

$$|\xi, f\rangle_e = \frac{1}{2} \frac{N_f^e}{N} (|\xi, f\rangle + |-\xi, f\rangle), \tag{37}$$

$$|\xi, f\rangle_o = \frac{1}{2} \frac{N_f^o}{N} (|\xi, f\rangle - |-\xi, f\rangle), \tag{38}$$

where

$$|\xi, f\rangle = N_f \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{n! f(n)!}} |n\rangle, \tag{39}$$

$$N_f \equiv N_f(|\xi|^2) = \left\{ \sum_{n=0}^{\infty} \frac{(|\xi|^2)^n}{n![f(n)!]^2} \right\}^{-1/2}, \tag{40}$$

$$N_f^e \equiv N_f^e(|\xi|^2) = \left\{ \sum_{n=0}^{\infty} \frac{(|\xi|^2)^{2n}}{(2n)![f(2n)!]^2} \right\}^{-1/2}, \tag{41}$$

$$N_f^o \equiv N_f^o(|\xi|^2) = \left\{ \sum_{n=0}^{\infty} \frac{(|\xi|^2)^{2n+1}}{(2n+1)![f(2n+1)!]^2} \right\}^{-1/2}. \tag{42}$$

Here, we find that the even (odd) nonlinear charge coherent states can also be generated from the states $(37) - (39)$ according to the following expression

$$|\xi, \underline{q}, f\rangle_{e(o)}$$

$$
= \begin{cases}
N_{\underline{q}f}^{e(o)} N_f^{-1}(|\xi_1|^2) \left[N_f^{e(o)}(|\xi_2|^2) \right]^{-1} \xi_1^{-\underline{q}} \int\limits_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{+\mathrm{i}\underline{q}\alpha} |e^{-\mathrm{i}\alpha}\xi_1, f\rangle \otimes |e^{\mathrm{i}\alpha}\xi_2, f\rangle_{e(o)}, & \underline{q} \geq 0, \\
N_{\underline{q}f}^{e(o)} N_f^{-1}(|\xi_1|^2) \left[N_f^{e(o)}(|\xi_2|^2) \right]^{-1} \xi_1^{+\underline{q}} \int\limits_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-\mathrm{i}\underline{q}\alpha} |e^{\mathrm{i}\alpha}\xi_2, f\rangle_{e(o)} \otimes |e^{-\mathrm{i}\alpha}\xi_1, f\rangle, & \underline{q} \leq 0,
\end{cases}
\tag{43}
$$

where $\xi = \xi_1\xi_2$. Such a representation is very useful since the properties of nonlinear coherent states and even (odd) nonlinear coherent states can now be employed in a study of the properties of even (odd) nonlinear charge coherent states. The expression for the latter given in (43) has a very simple group-theoretical interpretation: in (43) one suitably averages over the U(1)-group (caused by the charge operator Q) action on the product of nonlinear coherent states and even (odd) nonlinear coherent states, which then projects out the $Q = \underline{q}$ charge subspace contribution.

It is easy to see that in the two special cases of $f(n) = 1$ and $f(n) = \sqrt{[n]/n}$, the above discussion gives back the corresponding results for the usual even (odd) charge coherent states obtained in Ref. [44] and the even (odd) q-deformed charge coherent states in Ref. [64], respectively.

On the other hand, from the entangled nonorthogonal state point of view, the even (odd) nonlinear charge coherent states given in (43) are represented as a continuous entangled sum of nonlinear coherent states and even (odd) nonlinear coherent states. Therefore, following the definition of entangled coherent states [91−93], we call them generalized entangled nonlinear coherent states.

4. D-algebra realization of $\mathrm{SU}_f(1,1)$ generators

As is well known, the coherent state D-algebra [6,94] is a mapping of quantum observables onto a differential form that acts on the parameter space of coherent states, and has a beautiful application in the reformulation of the entire laser theory in terms of C-number differential equations [95]. We shall construct the D-algebra

realization of the f-deformed $\mathrm{SU}_f(1,1)$ generators corresponding to the unnormalized even and odd nonlinear charge coherent states, defined by

$$||\underline{q}\rangle_{e(o)} \equiv ||\xi, \underline{q}, f\rangle_{e(o)} = \left[N_{\underline{q}f}^{e(o)}\right]^{-1}|\xi, \underline{q}, f\rangle_{e(o)}. \tag{44}$$

Let $||\underline{q}\rangle$ denote a column vector composed of $||\underline{q}\rangle_e$ and $||\underline{q}\rangle_o$, i.e.,

$$||\underline{q}\rangle \equiv \left[\begin{array}{c} ||\underline{q}\rangle_e \\ ||\underline{q}\rangle_o \end{array} \right]. \tag{45}$$

The action of the operators A_i, $A_i^\dagger$, $\tilde{A}_i$, $\tilde{A}_i^\dagger$ and N_i on this column vector can be written in the matrix form of differential operators,

Positive Q Negative Q

$$A_1||\underline{q}\rangle = ||\underline{q}-1\rangle \qquad\qquad A_1||\underline{q}\rangle = \xi M||\underline{q}-1\rangle$$

$$A_2||\underline{q}\rangle = \xi M||\underline{q}+1\rangle \qquad\qquad A_2||\underline{q}\rangle = ||\underline{q}+1\rangle$$

$$A_1^\dagger||\underline{q}\rangle = \xi^{-\underline{q}}f^2(\tfrac{d}{d\xi}\xi)\tfrac{d}{d\xi}\xi^{\underline{q}+1}||\underline{q}+1\rangle \qquad A_1^\dagger||\underline{q}\rangle = f^2(\tfrac{d}{d\xi}\xi)\tfrac{d}{d\xi}M||\underline{q}+1\rangle$$

$$A_2^\dagger||\underline{q}\rangle = f^2(\tfrac{d}{d\xi}\xi)\tfrac{d}{d\xi}M||\underline{q}-1\rangle \qquad A_2^\dagger||\underline{q}\rangle = \xi^{\underline{q}}f^2(\tfrac{d}{d\xi}\xi)\tfrac{d}{d\xi}\xi^{-\underline{q}+1}||\underline{q}-1\rangle$$

$$N_1||\underline{q}\rangle = \left(\xi\tfrac{d}{d\xi}+\underline{q}\right)||\underline{q}\rangle \qquad N_1||\underline{q}\rangle = \xi\tfrac{d}{d\xi}||\underline{q}\rangle \tag{46}$$

$$N_2||\underline{q}\rangle = \xi\tfrac{d}{d\xi}||\underline{q}\rangle \qquad\qquad N_2||\underline{q}\rangle = \left(\xi\tfrac{d}{d\xi}-\underline{q}\right)||\underline{q}\rangle$$

$$\tilde{A}_1||\underline{q}\rangle = \xi^{-\underline{q}+1}\tfrac{1}{f^2(\tfrac{d}{d\xi}\xi)}\xi^{\underline{q}-1}||\underline{q}-1\rangle \qquad \tilde{A}_1||\underline{q}\rangle = \xi\tfrac{1}{f^2(\tfrac{d}{d\xi}\xi)}M||\underline{q}-1\rangle$$

$$\tilde{A}_2||\underline{q}\rangle = \xi\tfrac{1}{f^2(\tfrac{d}{d\xi}\xi)}M||\underline{q}+1\rangle \qquad \tilde{A}_2||\underline{q}\rangle = \xi^{\underline{q}+1}\tfrac{1}{f^2(\tfrac{d}{d\xi}\xi)}\xi^{-\underline{q}-1}||\underline{q}+1\rangle$$

$$\tilde{A}_1^\dagger||\underline{q}\rangle = \left(\xi\tfrac{d}{d\xi}+\underline{q}+1\right)||\underline{q}+1\rangle \qquad \tilde{A}_1^\dagger||\underline{q}\rangle = \tfrac{d}{d\xi}M||\underline{q}+1\rangle$$

$$\tilde{A}_2^\dagger||\underline{q}\rangle = \tfrac{d}{d\xi}M||\underline{q}-1\rangle \qquad \tilde{A}_2^\dagger||\underline{q}\rangle = \left(\xi\tfrac{d}{d\xi}-\underline{q}+1\right)||\underline{q}-1\rangle,$$

where

$$M = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \tag{47}$$

$\frac{1}{f(\frac{d}{d\xi}\xi)}$ is the inverse of $f(\frac{d}{d\xi}\xi)$, and the action of $f(\frac{d}{d\xi}\xi)$ on ξ^n is given by

$$f(\frac{d}{d\xi}\xi)\xi^n = f(n+1)\xi^n. \tag{48}$$

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Some useful relations for differential operators are as follows:

$$f(\frac{d}{d\xi}\xi)\frac{d}{d\xi} = \frac{d}{d\xi}f(\xi\frac{d}{d\xi}), \qquad \xi f(\frac{d}{d\xi}\xi) = f(\xi\frac{d}{d\xi})\xi, \tag{49}$$

$$\xi^{-n}f(\frac{d}{d\xi}\xi)\xi^n = f(\frac{d}{d\xi}\xi + n), \qquad \xi^{-n}f(\xi\frac{d}{d\xi})\xi^n = f(\xi\frac{d}{d\xi} + n), \tag{50}$$

where the action of $f(\xi\frac{d}{d\xi})$ is given by

$$f(\xi\frac{d}{d\xi})\xi^n = f(n)\xi^n. \tag{51}$$

Comparing the formulae in (46) in the q-deformed case of $f(n) = \sqrt{[n]/n}$ with those obtained in Ref. [64] for the even and odd q-deformed charge coherent states, and using (49) or (50), we have

$$\frac{d}{d_q\xi} = [\frac{d}{d\xi}\xi]\xi^{-1} = \xi^{-1}[\xi\frac{d}{d\xi}], \tag{52}$$

where $d/d_q\xi$ is a q-differential operator [49,54,96], defined by

$$\frac{d}{d_q\xi}f(\xi) = \frac{f(q\xi) - f(q^{-1}\xi)}{q\xi - q^{-1}\xi}. \tag{53}$$

Thus, we get an important result that the q-differential operator can be realized by the standard differential operator according to the expression (52). It should be noticed that the operator $\xi^{-1}[\xi\frac{d}{d\xi}]$ in (52) was already introduced by Solomon [72].

It is easy to check that in the two special cases of $f(n) = 1$ and $f(n) = \sqrt{[n]/n}$, the above discussion gives back that carried out in Refs. [44,64] for the usual even (odd) charge coherent states and the even (odd) q-deformed ones, respectively.

Let us define the f-deformed $\mathrm{SU}_f(1,1)$ algebra, which consists of three generators K_0, K_+ and K_-, where

$$K_- = A_1 A_2, \qquad K_+ = \tilde{A}_1^\dagger \tilde{A}_2^\dagger, \qquad K_0 = \frac{1}{2}(N_1 + N_2 + 1), \tag{54}$$

the latter satisfying the commutation relations

$$[K_+, K_-] = -2K_0, \qquad [K_0, K_\pm] = \pm K_\pm, \tag{55}$$

while their Hermitian conjugates satisfying the dual algebra

$$[K_-^\dagger, K_+^\dagger] = -2K_0, \qquad [K_0, K_\mp^\dagger] = \pm K_\mp^\dagger. \tag{56}$$

Note that K_0 is Hermitian whereas K_+ and K_- are not generally required to be Hermitian conjugate to each other. Obviously, this algebra ia a generalization of the SU(1,1) Lie algebra. When K_+ and K_- are Hermitian conjugate to each other in the special case of $f(N_i) = 1$, i.e., $K_-^\dagger = K_+$, the $\mathrm{SU}_f(1,1)$ algebra contracts to the SU(1,1) Lie algebra. Actually, the even and odd nonlinear charge coherent states are also eigenstates of the square of K_-.

The D-algebra of the $\mathrm{SU}_f(1,1)$ generators A may be defined for the action on the ket coherent states (45) or for that on the corresponding bras as

$$A||\underline{q}\rangle = D^k(A)||\underline{q}\rangle, \tag{57}$$

$$\langle\underline{q}||A = D^b(A)\langle\underline{q}||, \tag{58}$$

respectively. Using (46) and (54), we get for the former

$$D^k(K_-) = \xi M, \tag{59}$$

$$D^k(K_+) = \frac{d}{d\xi}(\xi\frac{d}{d\xi} + |\underline{q}|)M, \tag{60}$$

$$D^k(K_0) = \frac{1}{2}\left(2\xi\frac{d}{d\xi} + |\underline{q}| + 1\right)I, \tag{61}$$

$$D^k(K_-^\dagger) = \xi^{-|\underline{q}|}f^2(\frac{d}{d\xi}\xi)\frac{d}{d\xi}\xi^{|\underline{q}|+1}f^2(\frac{d}{d\xi}\xi)\frac{d}{d\xi}M, \tag{62}$$

$$D^k(K_+^\dagger) = \xi^{-|\underline{q}|+1}\frac{1}{f^2(\frac{d}{d\xi}\xi)}\xi^{|\underline{q}|}\frac{1}{f^2(\frac{d}{d\xi}\xi)}M, \tag{63}$$

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while the latter can be obtained from the adjoint relation

$$D^b(A) = \left[D^k(A^\dagger) \right]^*. \tag{64}$$

Thus, the D-algebra of the $\mathrm{SU}_f(1,1)$ generators corresponding to the unnormalized even and odd nonlinear charge coherent states has been realized in a differential-operator matrix form.

From (45), (47), (57) and (59), we clearly see that the unnormalized even and odd nonlinear charge coherent states can be transformed into each other by the action of the operator $A_1 A_2$. Actually, $A_1 A_2$ plays the role of a connecting operator between the two kinds of states.

5. Nonclassical properties of even and odd nonlinear charge coherent states

In this section, we will study some nonclassical properties of the even and odd nonlinear charge coherent states, such as $\mathrm{SU}_f(1,1)$ squeezing, single- or two-mode f-squeezing, and two-mode f-antibunching.

5.1. $\mathrm{SU}_f(1,1)$ squeezing

In analogy with the definition of $\mathrm{SU}_q(1,1)$ squeezing [64,97], which is a q-deformed analouge to $\mathrm{SU}(1,1)$ squeezing [44,98,99], we introduce $\mathrm{SU}_f(1,1)$ squeezing in terms of the Hermitian f-deformed quadrature operators

$$X_1 = \frac{K_-^\dagger + K_-}{2}, \qquad X_2 = \frac{\mathrm{i}(K_-^\dagger - K_-)}{2}, \tag{65}$$

which satisfy the commutation relation

$$[X_1, X_2] = \frac{\mathrm{i}}{2}[K_-, K_-^\dagger] = \frac{\mathrm{i}}{2}[(N_1+1)f^2(N_1+1)(N_2+1)f^2(N_2+1) - N_1 f^2(N_1)N_2 f^2(N_2)] \tag{66}$$

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and the uncertainty relation

$$\langle(\Delta X_1)^2\rangle\langle(\Delta X_2)^2\rangle \geq \frac{1}{16}|\langle[K_-, K_-^\dagger]\rangle|^2. \tag{67}$$

A state is said to be $SU_f(1,1)$ squeezed if

$$\langle(\Delta X_i)^2\rangle < \frac{1}{4}|\langle[K_-, K_-^\dagger]\rangle| \qquad (i = 1 \text{ or } 2). \tag{68}$$

Obviously, $SU(1,1)$ and $SU_q(1,1)$ squeezing are the two special cases of $SU_f(1,1)$ squeezing with $f(N_i) = 1$ and $f(N_i) = \sqrt{[N_i]/N_i}$, respectively. Therefore, $SU_f(1,1)$ squeezing is a natural extension of $SU(1,1)$ and $SU_q(1,1)$ squeezing.

Let us now calculate the fluctuations (variances) of X_1 and X_2 with respect to the even and odd nonlinear charge coherent states. Using (54), (57)−(59) and (64), we get

$$_e\langle\xi, \underline{q}, f|K_-^\dagger K_-|\xi, \underline{q}, f\rangle_e = |\xi|^2 \, \overline{\tanh}_{\underline{q}f}|\xi|^2, \tag{69}$$

$$_o\langle\xi, \underline{q}, f|K_-^\dagger K_-|\xi, \underline{q}, f\rangle_o = |\xi|^2 \, \overline{\coth}_{\underline{q}f}|\xi|^2, \tag{70}$$

where

$$\overline{\tanh}_{\underline{q}f}|\xi|^2 \equiv \frac{\overline{\sinh}_{\underline{q}f}|\xi|^2}{\overline{\cosh}_{\underline{q}f}|\xi|^2}, \qquad \overline{\coth}_{\underline{q}f}|\xi|^2 \equiv \frac{1}{\overline{\tanh}_{\underline{q}f}|\xi|^2}, \tag{71}$$

with

$$\overline{\sinh}_{\underline{q}f}x \equiv \left[N_{\underline{q}f}^o(x)\right]^{-2}, \qquad \overline{\cosh}_{\underline{q}f}x \equiv \left[N_{\underline{q}f}^e(x)\right]^{-2}. \tag{72}$$

Furthermore,

$$_{e(o)}\langle K_-\rangle_{e(o)} = {}_{e(o)}\langle K_-^\dagger\rangle_{e(o)} = 0. \tag{73}$$

Thus, the fluctuations are given by

$$_e\langle(\Delta X_1)^2\rangle_e = \frac{1}{4}{}_e\langle[K_-, K_-^\dagger]\rangle_e + \frac{1}{2}|\xi|^2(\cos 2\theta + \overline{\tanh}_{\underline{q}f}|\xi|^2), \tag{74}$$

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$$_e\langle(\Delta X_2)^2\rangle_e = \frac{1}{4}{}_e\langle[K_-, K_-^\dagger]\rangle_e + \frac{1}{2}|\xi|^2(-\cos 2\theta + \overline{\tanh}_{\underline{q}f}|\xi|^2), \qquad (75)$$

$$_o\langle(\Delta X_1)^2\rangle_o = \frac{1}{4}{}_o\langle[K_-, K_-^\dagger]\rangle_o + \frac{1}{2}|\xi|^2(\cos 2\theta + \overline{\coth}_{\underline{q}f}|\xi|^2), \qquad (76)$$

$$_o\langle(\Delta X_2)^2\rangle_o = \frac{1}{4}{}_o\langle[K_-, K_-^\dagger]\rangle_o + \frac{1}{2}|\xi|^2(-\cos 2\theta + \overline{\coth}_{\underline{q}f}|\xi|^2). \qquad (77)$$

According to (74)−(77), $\mathrm{SU}_f(1,1)$ squeezing of the states $|\xi, \underline{q}, f\rangle_{e(o)}$ occurs as long as

$$\pm\cos 2\theta + \overline{\tanh}_{\underline{q}f}|\xi|^2 < 0, \qquad (78)$$

$$\pm\cos 2\theta + \overline{\coth}_{\underline{q}f}|\xi|^2 < 0, \qquad (79)$$

for one of the sign choices. Choose θ to be $\frac{\pi}{2}$ or 0, so that $\pm\cos 2\theta = -1$. Conditions (78) and (79) may or may not be realized, depending on the assumed functional form of $f(n)$. When $|f(n)| \leq |f(n+1)|$ ($f(n) = 1$ and $f(n) = \sqrt{[n]/n}$ included), $\overline{\tanh}_{\underline{q}f}|\xi|^2 < 1$ for $|\xi| \leq f^2(1)$. Thus, condition (78) can be satisfied for $|\xi| \leq f^2(1)$ when $|f(n)| \leq |f(n+1)|$. When $f(n) = \frac{1}{\sqrt{n^{1-p}}}$ ($p \geq 1$) ($f(n) = 1$ included), from (18), (19), (71) and (72), we have

$$\overline{\coth}_{\underline{q}f}x = \frac{\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{[(2n)!(2n+|\underline{q}|)!]^p}}{\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{[(2n+1)!(2n+1+|\underline{q}|)!]^p}} \equiv \overline{\coth}_{\underline{q}p}x, \qquad (80)$$

where $x = |\xi|^2$. In fact, for arbitrary fixed values of $\underline{q}$ and p, there surely exists some range of x values such that $\overline{\coth}_{\underline{q}p}x < 1$. As shown in Ref. [44], it is indeed true for $p = 1$ (i.e., $f(n) = 1$). To make the above statement clear, we plot $\overline{\coth}_{\underline{q}p}x$ against x for various $\underline{q}$ and p in Fig. 1. Thus, for arbitrary fixed values of $\underline{q}$ and p, condition (79) can also be satisfied over some limited range of x values when $f(n) = \frac{1}{\sqrt{n^{1-p}}}$ ($p \geq 1$). Moreover, condition (79) also holds when $f(n) = \sqrt{[n]/n}$ [64].

From the above discussion, we know that the possibility of occurrence of $\mathrm{SU}_f(1,1)$ squeezing for the even and odd nonlinear charge coherent states, depends on the

particular form of $f(n)$. Indeed, the even nonlinear charge coherent states can exhibit $\mathrm{SU}_f(1,1)$ squeezing when $|f(n)| \leq |f(n+1)|$; the odd states can also do when $f(n) = \frac{1}{\sqrt{n^{1-p}}}$ ($p \geq 1$) and $f(n) = \sqrt{[n]/n}$. It is mentioned that for the usual even (odd) charge coherent states and the even (odd) q-deformed ones described by $f(n) = 1$ and $f(n) = \sqrt{[n]/n}$ respectively, a detailed discussion of SU(1,1) and $\mathrm{SU}_q(1,1)$ squeezing has been presented in Refs. [44,64].

It is easy to verify that the nonlinear charge coherent states satisfy the equality in (67) and that $\langle (\Delta X_1)^2 \rangle = \langle (\Delta X_2)^2 \rangle$. Therefore, the nonlinear charge coherent states, contrary to the even and odd ones, are not $\mathrm{SU}_f(1,1)$ squeezed.

5.2. Single-mode f-squeezing

In analogy with the definition of single-mode q-squeezing [64,97], which is a q-deformed analouge to single-mode squeezing [27,44,99], we introduce single-mode f-squeezing [82,83] in terms of the Hermitian f-deformed quadrature operators for the individual modes

$$Y_1 = \frac{A_1^\dagger + A_1}{2}, \qquad Y_2 = \frac{\mathrm{i}(A_1^\dagger - A_1)}{2},$$
$$Z_1 = \frac{A_2^\dagger + A_2}{2}, \qquad Z_2 = \frac{\mathrm{i}(A_2^\dagger - A_2)}{2}, \tag{81}$$

which satisfy the commutation relations

$$[Y_1, Y_2] = \frac{\mathrm{i}}{2}[A_1, A_1^\dagger], \qquad [Z_1, Z_2] = \frac{\mathrm{i}}{2}[A_2, A_2^\dagger], \tag{82}$$

and the uncertainty relations

$$\langle (\Delta Y_1)^2 \rangle \langle (\Delta Y_2)^2 \rangle \geq \frac{1}{16}|\langle [A_1, A_1^\dagger] \rangle|^2, \qquad \langle (\Delta Z_1)^2 \rangle \langle (\Delta Z_2)^2 \rangle \geq \frac{1}{16}|\langle [A_2, A_2^\dagger] \rangle|^2. \tag{83}$$

A state is said to be single-mode f-squeezed if

$$\langle (\Delta Y_i)^2 \rangle < \frac{1}{4}|\langle [A_1, A_1^\dagger] \rangle|, \qquad \langle (\Delta Z_i)^2 \rangle < \frac{1}{4}|\langle [A_2, A_2^\dagger] \rangle| \qquad (i = 1 \text{ or } 2). \tag{84}$$

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Obviously, single-mode squeezing and q-squeezing are the two special cases of single-mode f-squeezing with $f(N_i) = 1$ and $f(N_i) = \sqrt{[N_i]/N_i}$, respectively. Therefore, single-mode f-squeezing is a natural extension of single-mode squeezing and q-squeezing.

For the even and odd nonlinear charge coherent states, it always follows that

$$_{e(o)}\langle A_1 \rangle_{e(o)} = {}_{e(o)}\langle A_2 \rangle_{e(o)} = {}_{e(o)}\langle A_1^2 \rangle_{e(o)} = {}_{e(o)}\langle A_2^2 \rangle_{e(o)} = {}_{e(o)}\langle A_1^\dagger A_2 \rangle_{e(o)} = 0. \qquad (85)$$

Thus, the fluctuations are given by

$$\begin{aligned}
_{e(o)}\langle (\Delta Y_1)^2 \rangle_{e(o)} = {}_{e(o)}\langle (\Delta Y_2)^2 \rangle_{e(o)} &= \frac{1}{4}\left\{ {}_{e(o)}\langle [A_1, A_1^\dagger] \rangle_{e(o)} + 2\,{}_{e(o)}\langle A_1^\dagger A_1 \rangle_{e(o)} \right\} \\
&> \frac{1}{4}\,{}_{e(o)}\langle [A_1, A_1^\dagger] \rangle_{e(o)}, \qquad (86)
\end{aligned}$$

$$\begin{aligned}
_{e(o)}\langle (\Delta Z_1)^2 \rangle_{e(o)} = {}_{e(o)}\langle (\Delta Z_2)^2 \rangle_{e(o)} &= \frac{1}{4}\left\{ {}_{e(o)}\langle [A_2, A_2^\dagger] \rangle_{e(o)} + 2\,{}_{e(o)}\langle A_2^\dagger A_2 \rangle_{e(o)} \right\} \\
&> \frac{1}{4}\,{}_{e(o)}\langle [A_2, A_2^\dagger] \rangle_{e(o)}. \qquad (87)
\end{aligned}$$

This shows that there is no single-mode f-squeezing in both even and odd nonlinear charge coherent states. The same situation occurs for the nonlinear charge coherent states [100].

5.3. Two-mode f-squeezing

In analogy with the definition of two-mode q-squeezing [64,97], which is a q-deformed analouge to two-mode squeezing [44,99,101], we introduce two-mode f-squeezing [82,83] in terms of the Hermitian f-deformed quadrature operators for the two modes

$$W_1 = \frac{Y_1 + Z_1}{\sqrt{2}} = \frac{1}{\sqrt{8}}(A_1^\dagger + A_2^\dagger + A_1 + A_2), \quad W_2 = \frac{Y_2 + Z_2}{\sqrt{2}} = \frac{\mathrm{i}}{\sqrt{8}}(A_1^\dagger + A_2^\dagger - A_1 - A_2),$$

$$(88)$$

which satisfy the commutation relation

$$[W_1, W_2] = \frac{1}{4}i\left\{[A_1, A_1^\dagger] + [A_2, A_2^\dagger]\right\} \tag{89}$$

and the uncertainty relation

$$\langle(\Delta W_1)^2\rangle\langle(\Delta W_2)^2\rangle \geq \frac{1}{64}|\langle[A_1, A_1^\dagger]\rangle + \langle[A_2, A_2^\dagger]\rangle|^2. \tag{90}$$

A state is said to be two-mode q-squeezed if

$$\langle(\Delta W_i)^2\rangle < \frac{1}{8}|\langle[A_1, A_1^\dagger]\rangle + \langle[A_2, A_2^\dagger]\rangle| \qquad (i = 1 \text{ or } 2). \tag{91}$$

Obviously, two-mode squeezing and q-squeezing are the two special cases of two-mode f-squeezing with $f(N_i) = 1$ and $f(N_i) = \sqrt{[N_i]/N_i}$, respectively. Therefore, two-mode f-squeezing is a natural extension of two-mode squeezing and q-squeezing.

For the even and odd nonlinear charge coherent states, the fluctuations are given by

$$
\begin{aligned}
{}_{e(o)}\langle(\Delta W_1)^2\rangle_{e(o)} &= {}_{e(o)}\langle(\Delta W_2)^2\rangle_{e(o)} = \frac{1}{2}\left\{{}_{e(o)}\langle(\Delta Y_1)^2\rangle_{e(o)} + {}_{e(o)}\langle(\Delta Z_1)^2\rangle_{e(o)}\right\} \\
&= \frac{1}{2}\left\{{}_{e(o)}\langle(\Delta Y_2)^2\rangle_{e(o)} + {}_{e(o)}\langle(\Delta Z_2)^2\rangle_{e(o)}\right\} \\
&= \frac{1}{8}\left\{{}_{e(o)}\langle[A_1, A_1^\dagger]\rangle_{e(o)} + {}_{e(o)}\langle[A_2, A_2^\dagger]\rangle_{e(o)} + 2\,{}_{e(o)}\langle A_1^\dagger A_1\rangle_{e(o)}\right. \\
&\qquad \left. + 2\,{}_{e(o)}\langle A_2^\dagger A_2\rangle_{e(o)}\right\} \\
&> \frac{1}{8}\left\{{}_{e(o)}\langle[A_1, A_1^\dagger]\rangle_{e(o)} + {}_{e(o)}\langle[A_2, A_2^\dagger]\rangle_{e(o)}\right\}.
\end{aligned} \tag{92}
$$

This shows that there is no two-mode f-squeezing in both even and odd nonlinear charge coherent states. On the contrary, there is such f-squeezing in the nonlinear charge coherent states, depending on the particular form of $f(n)$. As shown in Refs. [44,64,100], two-mode f-squeezing does exist for the nonlinear charge coherent states with $f(n) = \frac{1}{\sqrt{n^{1-p}}}$ $(p \geq 1)$ and $f(n) = \sqrt{[n]/n}$.

5.4. Two-mode f-antibunching

In analogy with the definition of two-mode q-antibunching [64,97], which is a q-deformed analouge to two-mode antibunching [44,99], we introduce a two-mode f-correlation function as [82]

$$g^{(2)}(0) \equiv \frac{\langle (A_1^\dagger A_2^\dagger)^2 (A_1 A_2)^2 \rangle}{\langle A_1^\dagger A_2^\dagger A_1 A_2 \rangle^2} = \frac{\langle : (N_1 f^2(N_1) N_2 f^2(N_2))^2 : \rangle}{\langle N_1 f^2(N_1) N_2 f^2(N_2) \rangle^2} = \frac{\langle (K_-^\dagger)^2 K_-^2 \rangle}{\langle K_-^\dagger K_- \rangle^2}, \quad (93)$$

where A_i and $A_i^\dagger$ represent the annihilation and creation operators of f-deformed photons of a nonlinear light field for the ith mode and $::$ denotes normal ordering. Physically, $g^{(2)}(0)$ is a measure of f-deformed two-photon correlations in the nonlinear two-mode field and is related to the f-deformed two-photon number distributions. A state is said to be two-mode f-antibunched if

$$g^{(2)}(0) < 1. \quad (94)$$

Obviously, two-mode antibunching and q-antibunching are the two special cases of two-mode f-antibunching with $f(N_i) = 1$ and $f(N_i) = \sqrt{[N_i]/N_i}$, respectively. Therefore, two-mode f-antibunching is a natural extension of two-mode antibunching and q-antibunching.

For the even and odd nonlinear charge coherent states, we have

$$g_e^{(2)}(0) = \overline{\coth}_{qf}^2 |\xi|^2, \quad (95)$$

$$g_o^{(2)}(0) = \overline{\tanh}_{qf}^2 |\xi|^2. \quad (96)$$

From the above discussion about the function $\overline{\coth}_{qf} |\xi|^2$ $(\overline{\tanh}_{qf} |\xi|^2)$, we see that depending on the particular form of $f(n)$, $g_{e(o)}^{(2)}(0)$ can be less than 1 over some limited range of $|\xi|$ values, producing two-mode f-antibunching. Indeed, the even

nonlinear charge coherent states can display such f-antibunching when $f(n) = \frac{1}{\sqrt{n^{1-p}}}$ $(p \geq 1)$ and $f(n) = \sqrt{[n]/n}$; the odd states can also do when $|f(n)| \leq |f(n+1)|$. On the contrary, for the nonlinear charge coherent states we have $g^{(2)}(0) = 1$ so that no two-mode f-antibunching exists.

It can be shown that in the two special cases of $f(n) = 1$ and $f(n) = \sqrt{[n]/n}$, the nonclassical properties of the usual even (odd) charge coherent states studied in Ref. [44] and the even (odd) q-deformed ones in Ref. [64], are retrieved as expected, respectively.

6. Summary

Let us sum up the results obtained in the present paper.

(1) The (over)completeness has been proved of the even and odd nonlinear charge coherent states, defined as two orthonormalized eigenstates of both the square of the pair f-deformed annihilation operator $a_1 f(N_1) a_2 f(N_2)$ and the charge operator $Q = N_1 - N_2$.

(2) The even (odd) nonlinear charge coherent states have been shown to be generated by a suitable average over the $U(1)$-group (caused by the charge operator) action on the product of nonlinear coherent states and even (odd) nonlinear coherent states. They have also been demonstrated to be generalized entangled nonlinear coherent states.

(3) The D-algebra of the $\mathrm{SU}_f(1,1)$ generators corresponding to the even and odd nonlinear charge coherent states has been realized in a differential-operator matrix form.

(4) It has been shown that the even (odd) nonlinear charge coherent states have

the possibility of existence of the nonclassical properties in the particular form of $f(n)$, and exhibit $SU_f(1,1)$ squeezing for $|f(n)| \leq |f(n+1)|$ ($f(n) = \frac{1}{\sqrt{n^{1-p}}}$ ($p \geq 1$)) and two-mode f-antibunching for $f(n) = \frac{1}{\sqrt{n^{1-p}}}$ ($p \geq 1$) ($|f(n)| \leq |f(n+1)|$), but neither single-mode nor two-mode f-squeezing.

Acknowledgments

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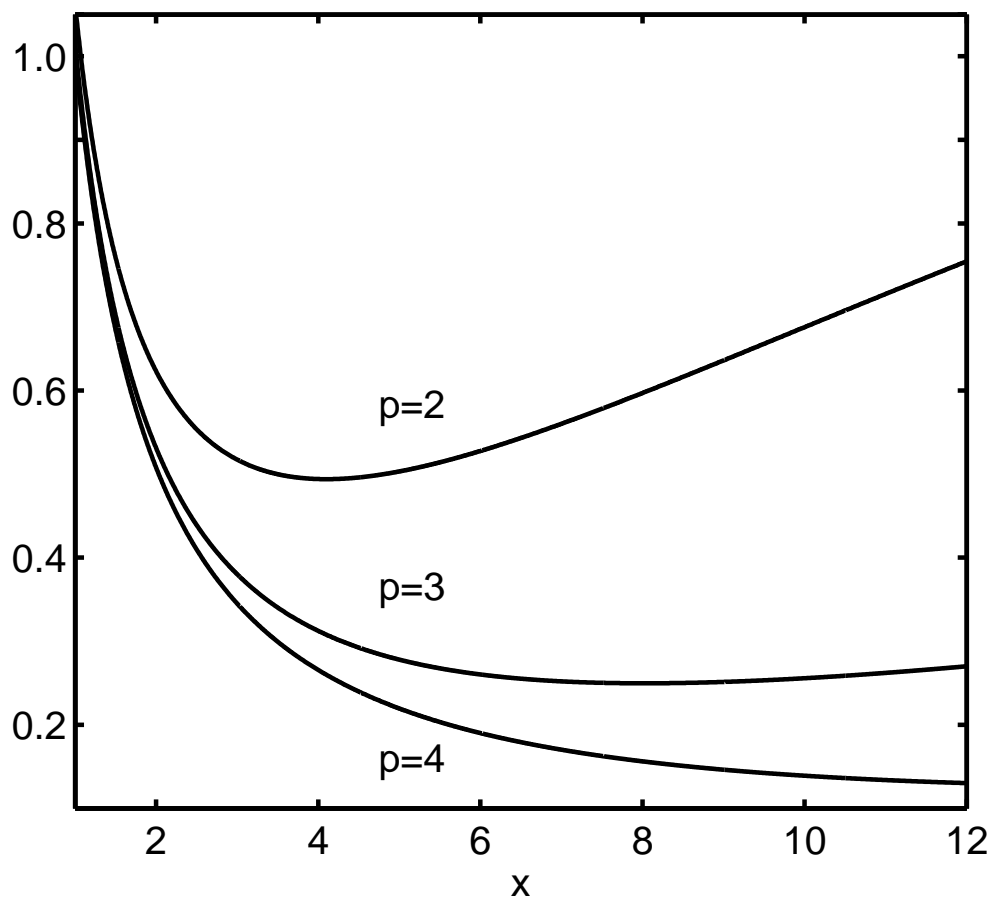
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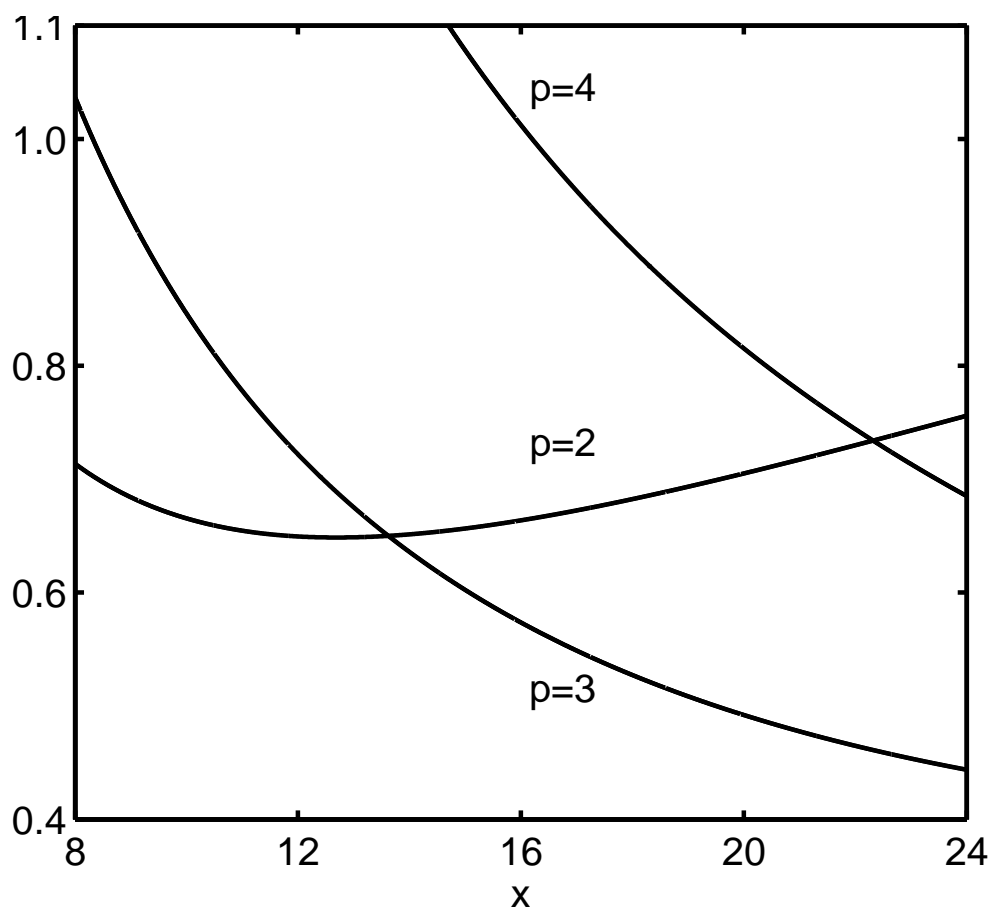[76] B. Roy and P. Roy, J. Opt. B 1 (1999) 341; 2 (2000) 65.

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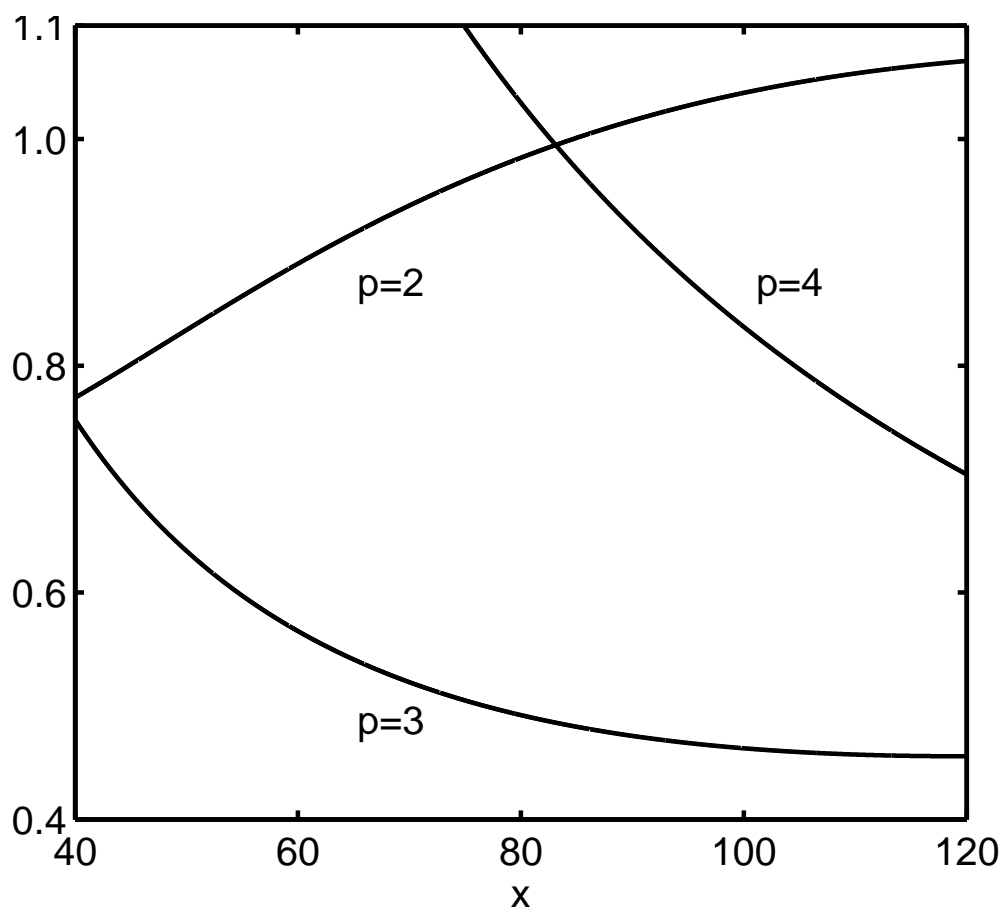[101] R. Loudon and P.L. Knight, J. Mod. Opt. 34 (1987) 709.

Figure caption

Fig. 1. $\overline{\coth}_{\underline{q}p}x$ against x for $p = 2,\ 3,\ 4$, with (a) $\underline{q} = 0$, (b) $\underline{q} = \pm 1$ and (c) $\underline{q} = \pm 2$.

(a)

(b)

(c)